

## CANONICAL EMBEDDINGS<sup>1</sup>

BY

J. MORROW AND H. ROSSI

**ABSTRACT.** In this paper the authors compare the embedding of a compact Riemann surface in its tangent bundle to the embedding as the diagonal in the product. These embeddings are proved to be first, but not second, order equivalent. The embedding of a hyperelliptic curve in its tangent bundle is described in an explicit way. Although it is not possible to be so explicit in the other cases, it is shown that in all cases, if the Riemann surface  $R$  has genus greater than two, then the blowdown of the zero section of the tangent bundle and the blowdown of the diagonal in the product have the same Hilbert polynomial.

**1. Introduction.** In [4], we discussed general facts about embeddings of compact manifolds, particularly in codimension 1 with a negative normal bundle (see §5). In this paper we shall be concerned with a very special case: that of a compact Riemann surface of genus  $g \geq 2$  into its tangent bundle (as the zero section). In this case the tangent bundle is negative and the zero section can be blown down (for terminology and notation, see [2]). We shall describe, as fully as we can, the analytic properties of the isolated singularity appearing at the blown-down zero section. There is a big difference between the hyperelliptic and nonhyperelliptic cases. For our purposes, a hyperelliptic Riemann surface is one which can be realized as a 2-sheeted cover of  $\mathbf{P}^1$  ([3, p. 247], see Definition 3.1 below). The function theory on such a surface can be made explicit, and as a result we can write down the equations defining the blown-down variety. In the nonhyperelliptic case this is not possible; however, Noether's theorem (Theorem 1.7) makes the situation much easier to deal with in a descriptive way (see Theorem 5.6 of [4]).

In this section we shall collect the necessary preliminary results, then in §2 we discuss the nonhyperelliptic case. In §§3–5 we deal with the hyperelliptic case in full detail. Finally in §6 we compare this embedding with the diagonal embedding in the Cartesian product; these embeddings are equivalent to first order (Theorem 4.2 of [4]).

First, we summarize some general remarks about the Grauert blowdown. Let  $X$  be an analytic space and  $Q$  a compact subvariety of  $X$ . Let  $\pi: X \rightarrow \tilde{X}$  be the quotient map identifying  $Q$  to a point, denoted  $q$ . We make  $\tilde{X}$  into a ringed space

---

Received by the editors September 19, 1979.

1980 *Mathematics Subject Classification*. Primary 32B30.

*Key words and phrases*. Grauert blow down, compact Riemann surface, tangent bundle, diagonal, tangent cone, canonical embedding, hyperelliptic, local ring, Noether's theorem, Hilbert polynomial.

<sup>1</sup>This research was partially supported by the National Science Foundation under grants MCS78-02139 (Morrow) and MCS78-08126 (Rossi).

© 1980 American Mathematical Society  
0002-9947/80/0000-0459/\$05.75

via the presheaf

$$\tilde{x}^\emptyset(U) = x^\emptyset(\pi^{-1}(U)), \quad U \text{ open in } X. \quad (1.1)$$

This makes  $\pi: X \rightarrow \tilde{X}$  a morphism.

1.2. DEFINITION.  $Q$  is *exceptional* in  $X$  if  $(\tilde{X}, \tilde{x}^\emptyset)$  is an analytic space.

If  $X$  is normal,  $\tilde{x}^\emptyset$  is a sheaf of germs of functions on  $\tilde{X}$ , and  $\tilde{X}$  is also normal. Note that, by definition

$$\tilde{x}^\emptyset_q = H^0(Q, x^\emptyset), \quad \tilde{x}m_q = H^0(Q, \mathcal{G})$$

where  $m_q$  is the maximal ideal at  $q$  and  $\mathcal{G}$  is the ideal sheaf of  $Q$  in  $X$ . We should not expect that

$$H^0(Q, \mathcal{G}^k) = \tilde{x}m_q^k, \quad k > 1$$

(see Proposition 6.5), so we cannot study the filtration of  $\tilde{x}Q_q$  by powers of the maximal ideal via the corresponding filtration on  $X$  along  $Q$ . Instead we have to use a result in [1]: Let  $\hat{m}^k$  be the ideal sheaf on  $X$  generated by  $\pi^{-1}m_q^k$  ( $\hat{m}^k = x^\emptyset \cdot \pi^{-1}m_q$ ). For  $\mathcal{F}$  a sheaf on  $X$ ,  $\pi_*\mathcal{F}$  is the direct image sheaf.

1.3. THEOREM [1, p. 113].  $\pi_*(\hat{m}^k)_q \subset m_q^{F(k)}$  where  $F(k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

REMARK. In general  $\mathcal{G}^k$  and  $\hat{m}^k$  are different. Let  $p_1$  and  $p_2$  be two points on a compact Riemann surface  $R$  such that  $H^0(R, (p_1 + p_2)) \cong \mathbb{C}$  (if  $R$  has genus 2 this is generically the case). Let  $\sigma$  generate  $H^0(R, (p_1 + p_2))$  and let  $r \in R$  be such that  $\sigma(r) \neq 0$ . Then  $H^0(R, (p_1 + p_2 - r)) = 0$ . For if  $\tau$  is the canonical section of  $(r)$ ,

$$H^0(R, (p_1 + p_2 - r)) \xrightarrow{\tau} H^0(R, (p_1 + p_2))$$

is injective. If  $\xi$  were a nonzero section of  $(p_1 + p_2 - r)$ ,  $\tau \cdot \xi$  would be a section of  $(p_1 + p_2)$  vanishing at  $r$  and thus not a multiple of  $\sigma$ .

Now  $(p_1 + p_2 - r)$  is a positive divisor, so if  $L$  is the line bundle of this divisor,  $L^{-1}$  is negative, and the zero section  $Q$  can be blown down to a point  $q$ . We have the exact sequence

$$0 \rightarrow H^0(L^{-1}, \mathcal{G}^2) \rightarrow H^0(L^{-1}, \mathcal{G}) \rightarrow H^0(R^1, \mathcal{G}/\mathcal{G}^2).$$

But  $\mathcal{G}/\mathcal{G}^2 = (p_1 + p_2 - r)$ , thus  $H^0(L^{-1}, \mathcal{G}) = m_q \cong H^0(L^{-1}, \mathcal{G}^2)$ , so  $\hat{m}_q \subset \mathcal{G}^2$  is distinct from  $\mathcal{G}$ .

Let  $R$  be a compact Riemann surface. We shall let  $T \rightarrow R$  denote the tangent bundle to  $R$  and  $K \rightarrow R$  its dual (the *canonical bundle* of  $R$ ). In general, if  $L \rightarrow R$  is any line bundle we shall let  $Z$  (or  $Z_L$  if the context requires that  $L$  be specified) represent the zero section of  $L$ . If the genus  $g$  of  $R$  is at least 2, then  $T$  is a negative bundle and its zero section can be blown down to a point in a normal analytic space  $V$  [2].  $K^n$  can be viewed as the sheaf of germs of holomorphic functions on  $T$  which are homogeneous of degree  $n$  on the fibers; these are the *n-fold differentials* on  $R$ . Thus  $H^0(R, K^n)$  ( $n \geq 0$ ) can be identified with a subspace of  $\mathcal{O}_{v,0}$ , the local ring of  $V$  at the blown-down zero section. Conversely, if  $f \in \mathcal{O}_{v,0}$ , we can lift  $f$  back to  $T$  and expand it in a Taylor series along  $Z$ . The terms in this Taylor series are in  $H^0(R, K^n)$ ,  $n \geq 0$ . Thus the ring  $\mathcal{K}(R) = \bigoplus_{n \geq 0} H^0(R, K^n)$  is contained in  $\mathcal{O}_{v,0}$  and every element of  $\mathcal{O}_{v,0}$  can be approximated (in any reasonable topology) by

$\mathcal{H}(R)$ . Thus  $\mathcal{H}(R)$  serves to determine the variety  $V$  (in fact, in the algebraic sense it is just the function ring of  $V$ ).

Let  $L \rightarrow R$  be any line bundle on  $R$ , and  $\sigma_0, \dots, \sigma_{n-1} \in H^0(R, L^*)$ . Since these are holomorphic functions on  $L$  which are linear on the fibers, the map  $F = (\sigma_0, \dots, \sigma_{n-1}): L \rightarrow \mathbb{C}^n$  maps  $L$  onto a set  $F(L)$  which is homogeneous (if  $z \in F(L)$ ,  $t \in \mathbb{C}$ , then  $tz \in F(L)$ ). If the  $\sigma$ 's have no base point (for each  $p \in R$ , there is a  $j$  such that  $\sigma_j(p) \neq 0$ ), then  $F$  is proper (since  $F^{-1}(0) = Z$  is compact) and  $F(L)$  is an analytic subvariety of  $\mathbb{C}^n$ . For  $z_0, \dots, z_{n-1}$ , coordinates for  $\mathbb{C}^n$ , let  $[z_0, \dots, z_{n-1}]$  represent the corresponding homogeneous coordinates for  $\mathbb{P}^{n-1}$ , and  $\pi: \mathbb{C}^n - \{0\} \rightarrow \mathbb{P}^{n-1}$  the projection map  $(z_0, \dots, z_{n-1}) \rightarrow [z_0, \dots, z_{n-1}]$ . Since  $\pi \circ F$  is constant on the fibers of  $L$  it defines a map  $[F]: R \rightarrow \mathbb{P}^{n-1}$  such that  $F(L) - \{0\} = \pi^{-1}([F](R))$  and  $F(L)$  is the cone over the projective variety  $[F](R)$ .

1.4. PROPOSITION.  $F: L - Z \rightarrow \mathbb{C}^n - \{0\}$  is an embedding if and only if  $[F]: R \rightarrow \mathbb{P}^{n-1}$  is an embedding.

PROOF. (We may replace  $R$  by any compact manifold.)  $L - Z$  and  $\mathbb{C}^n - \{0\}$  are fibered by  $\mathbb{C}^*$ ,  $F$  preserves the fibers, and  $[F]$  is the induced fiber map. Since  $F$  is linear and thus automatically biholomorphic on the fibers we easily see that  $F$  is injective if and only if  $[F]$  is injective.

Let  $p \in R$  and suppose  $\sigma_0(p) \neq 0$  (this is true for some  $\sigma$ ). Using inhomogeneous coordinates on  $\mathbb{P}^{n-1}$ :  $u_j = z_j z_0^{-1}$  ( $1 \leq j \leq n-1$ ), then the equations  $u_j = \sigma_j / \sigma_0$ ,  $1 \leq j \leq n-1$ , define the map  $[F]$ , and we can take  $z_0, u_1, \dots, u_{n-1}$  as coordinates for  $\mathbb{C}^n$ ; so  $F$  is defined by  $z_0 = \sigma_0$ ,  $u_j = \sigma_j \sigma_0^{-1}$ . Since  $d\sigma_0$  is not zero on the direction  $\{u_j = \text{constant}, 1 \leq j \leq n-1\}$ ,  $\text{rank}_\xi dF = \text{rank}_p d[F] + 1$  for  $\xi \neq 0$  in the fiber over  $p$ . Thus  $F$  has maximal rank at  $\xi$  if and only if  $[F]$  has maximal rank at  $p$ .

1.5. PROPOSITION. Suppose  $[F]: R \rightarrow \mathbb{P}^{n-1}$  is an embedding. If  $R$  is not a projective space and  $F$  linear, then  $V$  has a singularity at the vertex.

PROOF. If 0 is not a singular point of  $V$ , we find  $f_1, \dots, f_{n-d}$  ( $d = \dim V$ ) vanishing on  $V$  with  $df_1(0), \dots, df_{n-d}(0)$  independent. But since  $V$  is homogeneous these differentials also vanish on  $V$ . But  $\dim V = d$ , so  $V$  must coincide with the linear space on which these differentials vanish. Thus  $[F](R)$  is a linearly embedded projective space.

In this article we shall make a close study of the Grauert blowdown  $\tilde{T}$  of the tangent bundle  $T$  to a compact Riemann surface  $R$  of genus  $g > 2$ . Let  $w_0, \dots, w_{g-1}$  be a basis for  $H^0(R, K)$ , and consider the associated map  $F: T \rightarrow \mathbb{C}^g$ . Since there are no base points,  $[F]$  is well defined. We have to distinguish the cases where  $[F]$  is or is not an embedding. The following results are classical, and essential to our work. (See Proposition 4.2.)

1.6. THEOREM [3, pp. 247, 255]. If  $R$  is hyperelliptic,  $[F](R)$  is the normal rational curve parametrized by  $t \rightarrow [1, t, \dots, t^{g-1}]$  and  $[F]$  is a two-sheeted branched covering. If  $R$  is not hyperelliptic  $[F]$  is an embedding.

Part (1) of the following result is a theorem of Max Noether and (2) is due to Petri.

1.7. THEOREM [3, p. 253], [5], [6], [7]. *Let  $R$  be a nonhyperelliptic Riemann surface of genus  $g \geq 3$ , and let  $K$  be the canonical bundle of  $R$ . Let  $S^*H^0(K)$  be the symmetric algebra of  $H^0(K)$ , i.e.,  $S^*H^0(K) = \bigoplus_{n \geq 0} S^n H^0(K)$  with its ring structure.*

(1) *The canonical map  $\phi: S^*H^0(K) \rightarrow \bigoplus_{n \geq 0} H^0(K^n)$  is surjective.*

(2) *The kernel of  $\phi$  is generated by its elements of degree 2 and 3, except for  $g = 3$ , when it is generated by an element of degree four.*

**2. Nonhyperelliptic singularities.** Let  $R$  be a nonhyperelliptic Riemann surface of genus  $g \geq 3$ . Let  $\{w_0, \dots, w_{g-1}\}$  be a basis for  $H^0(R, K)$ , the space of holomorphic differentials of order 1. Then, as we have observed, (Theorem 1.2), the mapping  $[F] = [w_0, \dots, w_{g-1}]$  of  $R$  into  $\mathbf{P}^{g-1}$  is an embedding. Considered as a map  $F = (w_0, \dots, w_{g-1})$  of the tangent bundle  $T$  into  $\mathbf{C}^g$ ,  $F$  maps  $T$  onto the cone  $V = \pi^{-1}([F](R))$ , blowing the zero section down to the vertex (by Proposition 1.4). (Here  $\pi: \mathbf{C}^g - \{0\} \rightarrow \mathbf{P}^{g-1}$  is the defining map.)  $T \rightarrow R$  is the pull-back via  $[F]$  of the tautological bundle on  $\mathbf{P}^{g-1}$  and therefore  $F: T \rightarrow V$  is the quadratic transform of the origin of  $V$ . By Noether's theorem the hypotheses of Proposition 5.6 of [4] are satisfied, so  $V$  is normal. Summarizing we have

2.1. LEMMA. *Let  $R$  be a nonhyperelliptic Riemann surface. The image  $V$  of the tangent bundle  $T$  via a basis of  $H^0(K)$  is an analytic subvariety of  $\mathbf{C}^g$  with a normal isolated singularity at the origin.  $T$  is the quadratic transform of  $V$ .*

Furthermore, Theorem 1.7 applies immediately to give

2.2. THEOREM. (i)  $g = 3$ .  $V$  is a hypersurface in  $\mathbf{C}^3$  defined by a homogeneous polynomial of degree 4.

(ii)  $g > 3$ .  $V$  is the zero locus in  $\mathbf{C}^g$  of a set of homogeneous polynomials of degree 2 and 3.

In either case,  $V$  is the cone over the canonical curve  $[F](R)$ .

2.3. PROPOSITION. *The multiplicity  $\mu_0(V)$  of  $V$  at the origin is  $2g - 2$ .*

PROOF. Since  $V$  is the cone over the curve  $C = [F](R)$  in  $\mathbf{P}^{g-1}$ ,  $\mu_0(V) = \deg C$ . Let

$$H = \left\{ \sum_{j=0}^{g-1} a_j z_j = 0 \right\}$$

be a hyperplane in  $\mathbf{P}^{g-1}$ . By Bezout's theorem [3, p. 670]  $\deg C = (C, H)$  where  $(C, H)$  is the intersection multiplicity. But  $(C, H)$  is the degree of the divisor  $\sum_{j=0}^{g-1} a_j w_j$  on  $R$ . Since  $\sum a_j w_j$  is a holomorphic one-form on  $R$  it has degree  $2g - 2$ , unless it is identically zero. But, since the  $w_j$  are independent,  $\sum a_j w_j = 0$  only when all the  $a_j$  are zero.

2.4. PROPOSITION. *The minimal embedding dimension of  $V$  at 0 is  $g$ .*

PROOF. Suppose  $f$  is holomorphic in a neighborhood of 0, and vanishes identically on  $V$ . Let  $df(0) = \sum_{j=0}^{g-1} a_j z_j$ . Since  $V$  is a cone,  $df(0)$  vanishes identically on  $V$  also. But  $df(0) \circ F = \sum a_j w_j \equiv 0$  implies  $a_0 = \cdots = a_{g-1} = 0$ , so  $df(0) = 0$ . Thus  $V$  is already minimally embedded in  $\mathbb{C}^g$ .

Finally, our last observation holds for the Grauert blow down of  $T$  in both the hyperelliptic and nonhyperelliptic cases.

2.5. PROPOSITION.  $V$  is Gorenstein.

PROOF. By definition, we have to produce a global nowhere-vanishing 2-form on  $V - \{0\} = T - Z$ . Let  $\{U_\alpha\}$  be a system of local trivializations of  $T$  with coordinates  $z_\alpha, t_\alpha$  ( $t_\alpha$  is the fiber coordinate). The transition functions are

$$z_\alpha = f_{\alpha\beta}(z_\beta), \quad t_\alpha = f'_{\alpha\beta}(z_\beta)t_\beta.$$

We easily compute that

$$\frac{dz_\alpha \wedge dt_\alpha}{t_\alpha^2} = \frac{dz_\beta \wedge dt_\beta}{t_\beta^2}$$

so the global form is given by

$$w = \frac{dz_\alpha \wedge dt_\alpha}{t_\alpha^2} \text{ in } U_\alpha.$$

**3. The hyperelliptic case: the algebra of holomorphic differentials.** We continue to study the tangent bundle  $T$  of a (nonsingular) curve  $R$  of genus  $g \geq 2$ . However, for the next few sections we shall deal only with the case where  $R$  is hyperelliptic.

3.1. DEFINITION [3, pp. 247, 255].  $R$  is *hyperelliptic* if  $R$  can be realized as the desingularization of the curve in  $\mathbb{C}^2$  with equation

$$\rho(u, z) = u^2 - \prod_{j=1}^{2g+1} (z - e_j) = 0 \quad (3.2)$$

(with points over infinity adjoined).

In (3.2) the  $e_j$ ,  $1 \leq j \leq 2g+1$ , are distinct nonzero complex numbers (the exclusion of 0 is a convenience and imposes no restrictions).

We consider  $u$  and  $z$  to be meromorphic functions on  $R$  satisfying the relation (3.2) and generating the field of rational functions on  $R$ . We shall now compute the algebra  $K(R) = \bigoplus_{n \geq 0} H^0(K^n)$  of holomorphic differentials in terms of  $z$  and  $u$ . By the Riemann-Roch theorem

$$\dim H^0(K^n) = \begin{cases} g & \text{if } n = 1, \\ (2n-1)(g-1) & \text{if } n \geq 2. \end{cases} \quad (3.3)$$

For each  $j$

$$w_j = z^j dz / u \quad (3.4)$$

is a meromorphic differential on  $R$ .

3.5. PROPOSITION.  $w_0, \dots, w_{g-1}$  are holomorphic differentials and give a basis for  $H^0(K)$ .

PROOF. At finite points where  $u \neq 0$ , clearly the  $w_j$  are holomorphic. Let  $(0, e_p) \in R$ . At such a point  $u$  is a local coordinate since

$$\frac{\partial p}{\partial z}(0, e_p) = - \prod_{k \neq p} (e_p - e_k) \neq 0.$$

Call this number  $d_p$ . Then, in terms of  $u$

$$z = e_p - d_p^{-1}u^2 + \cdots, \quad dz = -2d_p^{-1}u du + \cdots, \quad (3.6)$$

where the ellipsis refers to higher powers of  $u$ . Thus  $u^{-1} dz$  (and with it all the  $w_j$ ) is holomorphic at  $(0, e_p)$ .

At  $\infty$  we take a local parameter  $t$  so that  $z = t^{-2}$ . Then, in terms of  $t$ ,  $u$  has a zero of order  $2g + 1$ , so

$$u(t) = v(t)t^{-(2g+1)} \quad (3.7)$$

with  $v(0) \neq 0$ . Then

$$u^{-1} dz = -2v^{-1}t^{2g-2} dt, \quad w_j = -2v^{-1}t^{2(g-1-j)} dt, \quad (3.8)$$

so  $w_j$  is holomorphic as long as  $0 \leq j \leq g - 1$ . Clearly these  $w_j$  are independent (over  $\mathbb{C}$ ) so form a basis for  $H^0(K)$ .

We turn now to  $H^0(K^n)$  for  $n \geq 2$ . The differential

$$\theta = z^i u^j (dz/u)^n, \quad 0 \leq i, j,$$

is holomorphic over finite points since  $u^{-1} dz$  is. At  $\infty$ , computing as above, we find

$$\theta = (-2)^n v^{j-n} t^e dt,$$

where  $e = 2n(g - 1) - (2i + 2gj + j)$ . Since  $v$  is invertible, the differential  $\theta$  is holomorphic if and only if

$$2i + 2gj + j \leq 2n(g - 1). \quad (3.9)$$

We can select  $i$  and  $j$  so as to give basis for  $H^0(K^n)$ ; to do that carefully we consider different cases.

$$g = 2, \quad n = 2. \quad \{z^i (dz/u)^2, 0 \leq i \leq 2\} \text{ are a basis.} \quad (3.10)$$

In this case (3.9) becomes  $2i + 5j \leq 4$ , so these are the only allowable values of  $i$  and  $j$ . A computation at  $\infty$  shows these to be independent.

$$g = 2, \quad n > 2. \quad \begin{cases} z^i (dz/u)^n, & 0 \leq i \leq n, \\ z^i u (dz/u)^n, & 0 \leq i \leq n - 3, \end{cases} \quad (3.11)$$

are a basis.

Here (3.9) becomes  $2i + 5j \leq 2n$ . Taking  $j = 0, 1$  we obtain the indicated ranges of  $i$ . Since this gives us  $2n - 1$  differentials, we need only show that they are independent. Computing at  $\infty$ , the expression (3.11) becomes

$$\begin{aligned} k_i(t) t^{2(n-i)} dt, & \quad 0 \leq i \leq n, \\ h_i(t) t^{2(n-i)-5} dt, & \quad 0 \leq i \leq n - 3, \end{aligned}$$

with the  $k_i, h_i$  nonvanishing. Since the powers of  $t$  are all distinct, these are independent.

$$g > 2, \quad n \geq 2. \quad \begin{cases} z^i(dz/u)^n, & 0 \leq i \leq n(g-1), \\ z^i u(dz/u)^n, & 0 \leq i \leq (n-2)(g-1) + (g-3), \end{cases} \quad (3.12)$$

are a basis.

These are  $(2n-1)(g-1)$  forms satisfying the condition (3.9), so are in  $H^0(K^n)$ . Again, we can demonstrate their independence by calculating at  $\infty$  and comparing the exponents of  $t$ .

Thus (3.9)–(3.12) gives us a basis for each  $H^0(K^n)$ . We consider now the algebraic structure of the ring  $K(R) = \bigoplus_{n \geq 0} H^0(K^n)$ . We shall describe a system of (algebraic) generators and then compute the relations.

For  $g = 2$ , let

$$\sigma_0 = u(dz/u)^3. \quad (3.13)$$

For  $g > 2$ , let

$$\tau_j = z^j u(dz/u)^2, \quad 0 \leq j \leq g-3. \quad (3.14)$$

Recall also the first-order forms  $w_0, \dots, w_{g-1}$  defined by (3.4).

3.15. THEOREM. (I)  $g = 2$ .

(i) Every element of  $H^0(K^2)$  is a quadratic polynomial in  $w_0, w_1$ , hence  $S^2 H^0(K) \rightarrow H^0(K^2)$  is surjective;

(ii)  $S^3 H^0(K) \rightarrow H^0(K^3)$  is not surjective; however

(iii) for  $n \geq 3$ , every element of  $H^0(K^n)$  is a polynomial in  $w_0, w_1, \sigma_0$ ; this polynomial is a sum of homogeneous terms of degree 1 or 0 in  $\sigma_0$  and correspondingly of degree  $n-3$  or  $n$  in  $w_0, w_1$ . Thus

(iv) for  $n \geq 3$ ,  $S^{n-3} H^0(K) \otimes H^0(K^3) \rightarrow H^0(K^n)$  is surjective.

(II)  $g > 2$ .

(i)  $S^2 H^0(K) \rightarrow H^0(K^2)$  is not surjective; however

(ii) for  $n \geq 2$ , every element of  $H^0(K^n)$  is a polynomial in  $w_0, \dots, w_{g-1}, \tau_0, \dots, \tau_{g-3}$ ; this polynomial is a sum of homogeneous terms of degree 1 or 0 in the  $\tau_j$  and correspondingly of degree  $n$  or  $n-2$  in the  $w_i$ . Thus

(iii) for  $n \geq 2$ ,  $S^{n-2} H^0(K) \otimes H^0(K^2) \rightarrow H^0(K^n)$  is surjective.

PROOF. (I)(i) The basis (3.10) can be written as  $\{w_0^2, w_0 w_1, w_1^2\}$ . Thus (i) is proven.

(ii)  $\sigma_0$  is not a polynomial in  $w_0, w_1$  since  $dz$  and  $u^{-1}$  do not appear to the same power in  $\sigma_0$ .

(iii) To show the surjectivity in orders  $n > 2$ , it suffices to write the basis elements (3.11) as monomials in  $w_0, w_1, \sigma_0$  of degree 1 or 0 in  $\sigma_0$ . This clearly can be done

$$z^i(dz/u)^n = w_0^{n-i} w_1^i \quad (i \leq n); \quad z^i u(dz/u)^n = w_0^{n-i-3} w_1^i \sigma_0 \quad (1 \leq n-3).$$

Thus (I) is proven.

(II)(i) No  $\tau_j$  is a polynomial in the  $w_i$  since  $dz$  and  $u^{-1}$  do not appear to the same power in  $\tau_j$ .

(ii) We show that the basis elements (3.12) are monomials in the  $w_i, \tau_j$  of degree 1 or 0 in  $\tau_j$ . Given  $i$  and  $n$  with  $0 \leq i \leq n(g-1)$ , write  $i = k(g-1) + r$  with  $k < n$  and  $0 < r \leq g-1$ . Then

$$z^i (dz/u)^n = (z^{g-1} dz/u)^k (z^r dz/u) (dz/u)^{n-k-1} = w_{g-1}^k w_r w_0^{n-k-1}. \quad (3.16)$$

Now, if  $i \leq (n-2)(g-1) + (g-3)$  we can write

$$z^i u (dz/u)^n = [z^j u (dz/u)^2] [z^k (dz/u)^{n-2}]$$

with  $j \leq g-3$  and  $k \leq g-1$ . The first factor is just  $\tau_j$  and the second factor can be handled as above. Thus the theorem is proven.

This theorem shows that Noether's theorem (1.7) is not true in the hyperelliptic case, and gives the appropriate version. We will describe the analogue of Petri's theorem in §5.

**4. The embedding in Euclidean space.** Let  $T$  be the tangent bundle to  $R$ . In this section we shall consider the  $n$ th order differentials on  $R$  explicitly as holomorphic functions on  $T$  which are homogeneous of order  $n$  on the fibers. Let

$$\begin{cases} F_0: T \rightarrow \mathbb{C}^g: F_0 = (w_0, \dots, w_{g-1}), \\ F: T \rightarrow \mathbb{C}^3: F = (w_0, w_1, \sigma_0) \quad \text{if } g = 2, \\ F: T \rightarrow \mathbb{C}^{2g-2}: F = (w_0, \dots, w_g, \tau_0, \dots, \tau_{g-3}) \quad \text{if } g > 2. \end{cases} \quad (4.1)$$

**4.2. PROPOSITION.** *The canonical map  $F_0$  takes  $T$  onto the cone  $V_0$  over the rational curve in  $P^{g-1}$  parametrized by  $t \rightarrow [1, t, \dots, t^{g-1}]$ .  $F_0$  maps  $Z$  down to the vertex, and is a 2-sheeted branched covering map on  $T - Z$ .*

**PROOF.** At finite points  $(z, u, \xi)$ , where  $\xi$  is the fiber coordinate,

$$F_0(z, u, \xi) = \left( \frac{dz(\xi)}{u}, \frac{z dz(\xi)}{u}, \dots, \frac{z^{g-1} dz(\xi)}{u} \right).$$

Obviously, the image is dense in the cone  $V_0$ , so  $F_0$  maps all of  $T$  onto  $V_0$ .  $F_0(z, u, \xi) = F_0(z', u', \xi')$  if and only if  $z = z'$  and  $(u, \xi) = \pm (u', \xi')$ , so the map is strictly 2 : 1 where  $\xi \neq 0$  and  $u \neq 0$ . This calculation holds at  $\infty$  as well.

We now observe that the image  $V$  of  $F$  is precisely the Grauert blowdown of  $T$ . Then we shall compute the equations defining  $V$ , and in the next section conclude that the image  $V_0$  of  $F_0$  is (essentially) the tangent cone to  $V$ .

**4.3. THEOREM.**  *$F: T \rightarrow V$  is the Grauert blowdown of  $T$ .  $T$  is obtained from  $V$  by monoidal transform of  $\mathfrak{m}^3$  ( $\mathfrak{m}^2$ ) if  $g = 2$  ( $g > 2$ ) ( $\mathfrak{m}$  is the maximal ideal of  $V$  at 0).*

**PROOF.** The results of §3 tell us that the coordinates of  $F$ , as functions on  $T$ , generate the algebra  $\bigoplus_{n \geq 0} H^0(R, T^{*n})$ . Thus the first statement follows from Theorem 5.6 of [4]. The second statement follows from Theorem 5.5 of [4].

We shall describe the quadratic transform (the monoidal transform of  $\mathfrak{m}$ ) after computing the equations defining  $V$ . These are the relations among the holomorphic differentials, and (as we shall see) we need only consider fourth order.



We consider first the case  $g = 2$ . Then  $V$  is a subvariety of  $\mathbb{C}^3$ . Let  $x_0, x_1, y$  be the coordinates of  $\mathbb{C}^3$  so that  $F$  is given by  $x_0 = w_0, x_1 = w_1, y = \sigma_0$ .

4.4. THEOREM. ( $g = 2$ ).  $V$  is given by the equation

$$y^2 = Q(x_0, x_1) = x_0 x_1^5 + a_1 x_0^2 x_1^4 + \cdots + a_1 x_0^5 x_1 + x_0 x_0^6 \quad (4.5)$$

where the defining equation of  $R$  is

$$u^2 = \prod_{j=1}^5 (z - e_j) = z^5 + a_4 z^4 + \cdots + a_0. \quad (4.6)$$

PROOF. We can view  $z, u, dz$  as meromorphic functions on  $T$ , and  $x_0, x_1, y$  as holomorphic functions. We have the relations (4.6) and

$$x_0 = u^{-1} dz, \quad x_1 = zu^{-1} dz, \quad y = u^{-2}(dz)^3.$$

Hence

$$\begin{aligned} Q(x_0, x_1) &= (z^5 + Q_4 z^4 + \cdots + a_1 z + a_0)(u^{-1} dz)^6 \\ &= u^2(u^{-1} dz)^6 = (u^{-2}(dz)^3)^2 = y^2. \end{aligned}$$

Thus (4.5) is satisfied on  $V$ . Since the  $e_j$  are all distinct,  $Q$  is not a perfect square, so  $y^2 - Q$  is irreducible. Since  $V$  is a hypersurface contained in the irreducible hypersurface  $y^2 - Q = 0$ , they must coincide.

REMARK. We can conclude that the relation  $\sigma_0^2 = Q(w_0, w_1)$  generates all the relations among the holomorphic forms.

Now we turn to the case of higher genus. Let  $x_0, \dots, x_{g-1}, y_0, \dots, y_{g-3}$  be coordinates of  $\mathbb{C}^3$  so that  $F$  is given by  $x_i = w_i, y_j = t_j$ .

4.7. THEOREM. ( $g > 2$ ).  $V$  is given by the equations

$$\begin{cases} x_0 x_2 = x_1^2, \\ x_1 x_3 = x_2^2, \\ \vdots \\ x_{g-3} x_{g-1} = x_{g-2}^2, \\ x_0 x_{g-1} = x_1 x_{g-2}, \end{cases} \quad (4.8)$$

$$\begin{cases} y_0 y_2 = y_1^2, \\ y_1 y_3 = y_2^2, \\ \vdots \\ y_{g-5} y_{g-3} = y_{g-4}^2, \end{cases} \quad (4.9)$$

$$y_j x_0 = y_0 x_j, \quad 0 \leq j \leq g-3, \quad (4.10)$$

$$y_j^2 = Q_j(x_0, \dots, x_{g-1}), \quad 0 \leq j \leq g-3, \quad (4.11)$$

where the  $Q_j$  are homogeneous polynomials of degree 4, not containing terms of the form  $cx_{g-1}^4$ .

PROOF. First we describe the functions  $Q_j$ ,  $0 \leq j \leq g-3$ . Write the defining equation of  $R$  as

$$u^2 = \prod_{j=1}^{2g+1} (z - e_j) = z^{2g+1} + a_{2g}z^{2g} + \cdots + a_1z + a_0. \quad (4.12)$$

In the following, we shall be using the standard shorthand for monomials

$$X^k = X_1^{k_1} \cdots X_n^{k_n} \quad \text{where } k = (k_1, \dots, k_n).$$

It turns out that the functions  $Q$  are defined by the relations among the monomials of degree 4. For each  $i \leq 4(g-1)$ , let  $i = k(g-1) + r$  where  $k < 4$  and  $0 < r \leq g-1$  and define  $\alpha(i)$  by the expression (3.16):

$$z^i(u^{-1} dz)^n = w^{\alpha(i)}. \quad (4.13)$$

Then for  $j \leq g-3$  take

$$Q_j(x_0, \dots, x_{g-1}) = x^{\alpha(2j+2g+1)} + a_{2g}x^{\alpha(2j+2g)} + \cdots + a_1x^{\alpha(2j+1)} + a_0x^{\alpha(2j)},$$

where  $a_0, \dots, a_{2g}$  are defined by (4.12).

Let  $W$  be the variety defined by the equations (4.8)–(4.11). First, we verify that  $V$  satisfies these equations, i.e.,  $V \subset W$ . Recall that  $V$  is given as the image of  $F$  defined by  $x_i = w_i, y_j = \tau_j$  where

$$w_i = z^i(u^{-1} dz), \quad \tau_j = z^j u(u^{-1} dz)^2.$$

It is immediate that (4.8)–(4.10) are satisfied. To verify (4.11) we use (4.12) as follows.

$$\begin{aligned} \tau_j^2 &= z^{2j} u^2 (u^{-1} dz)^4 = z^{2j} (z^{2g+1} + a_{2g}z^{2g} + \cdots + a_1z + a_0) (u^{-1} dz)^4 \\ &= w^{\alpha(2j+2g+1)} + a_{2g}w^{\alpha(2j+2g)} + \cdots + a_1w^{\alpha(2j+1)} + a_0w^{\alpha(2j)} \end{aligned}$$

using (4.13). Thus  $y_j^2 = Q_j(x)$ , verifying equation (4.11).

Now we verify that  $W \subset V$ . Let  $q = (x_0, \dots, x_{g-1}, y_0, \dots, y_{g-3})$  be in  $W$ . We have to proceed by cases.

(A)  $x_0x_1y_0 \neq 0$ . By equations (4.8)–(4.10), all the coordinates are nonzero, and we can express them all in terms of  $x_0, x_1, y_0$ :

$$q = \left( x_0, x_1, \frac{x_1^2}{x_0}, \frac{x_1^3}{x_0^2}, \dots, \frac{x_1^{g-1}}{x_0^{g-2}}, y_0, y_0 \frac{x_1}{x_0}, y_0 \left( \frac{x_1}{x_0} \right)^2, \dots, y_0 \left( \frac{x_1}{x_0} \right)^{g-3} \right). \quad (4.14)$$

Let  $z = x_1x_0^{-1}$  and  $u = y_0x_0^{-2}$ . By (4.11)

$$\begin{aligned} y_0^2 &= Q_0(x_0, \dots, x_{g-1}) = x_0^4 Q_0(1, z, z^2, \dots, z^{g-1}) \\ &= x_0^4 (z^{2g+1} + a_{2g}z^{2g} + a_1z + a_0) = x_0^4 \prod_{j=1}^{2g+1} (z - e_j). \end{aligned}$$

Thus

$$u^2 = y_0^2 x_0^{-4} = \prod_{j=1}^{2g+1} (z - e_j),$$

so  $(z, u) \in R$ . Let  $\xi$  be the point in  $T$  lying above  $(z, u)$  at which  $dz(\xi) = y_0 x_0^{-1}$ . Then

$$F(\xi) = \left( \frac{dz(\xi)}{u}, \dots, \frac{z^{g-1} dz(\xi)}{u}, \frac{dz(\xi)^2}{u}, \dots, \frac{z^{g-3} dz(\xi)^2}{u} \right) = q$$

(by resubstituting  $z = x_1 x_0^{-1}$ ,  $u = y_0 x_0^{-2}$ ,  $dz(\xi) = y_0 x_0^{-1}$ ). Thus  $q \in V$ .

(B)  $x_0 \neq 0$ ,  $x_1 = 0$ . Then, by (4.8),  $x_i = 0$  for  $i \geq 2$ . Thus  $y_0^2 = Q_0(x_0, 0, \dots, 0) = a_0 x_0^4$ , where  $a_0 = -\prod e_j \neq 0$ . Thus also  $y_0 \neq 0$  and by (4.9),  $y_j = 0$  for  $j \geq 1$ , so that  $q = (x_0, 0, \dots, 0, y_0, 0, \dots, 0)$ . Let  $u = y_0 x_0^{-2}$ . Then

$$u^2 = y_0^2 x_0^{-4} = a_0 = \prod_{j=0}^{2g+1} (0 - e_j),$$

so  $(0, u) \in R$ . Choose  $\xi$  above  $(0, u)$  again so that  $dz(\xi) = y_0 x_0^{-1}$  and calculate as above that  $F(\xi) = q$ .

(C)  $x_0 \neq 0$ ,  $x_1 \neq 0$ , but  $y_0 = 0$ . Then, by (4.8) no  $x_i$  is zero and  $x_i = x_1^i x_0^{i-1}$ ,  $i \geq 2$ . Further, by (4.10) all  $y_j = 0$ . By (4.11),  $Q_0(x_0, \dots, x_{g-1}) = 0$ . Let  $z = x_1 x_0^{-1}$ . Then

$$0 = x_0^{-4} Q_0(x_0, \dots, x_{g-1}) = Q_0(1, z, \dots, z^{g-1}) = \prod_{j=1}^{2g+1} (z - e_j),$$

so  $z$  is one of the roots, say  $e_p$ . Then  $(z, 0) \in R$  and by (3.6), at  $(z, 0)$   $dz/u = -2d_p^{-1} du$ , where  $d_p = -\prod_{k \neq p} (e_p - e_k)$ . Choose  $\xi$  in  $T$  above  $(z, 0)$  so that  $du(\xi) = -\frac{1}{2} d_p x_0$ . Then  $F(\xi) = (x, 0) = q$ .

(D) Finally, suppose  $x_0 = 0$ . Then all  $x_i = 0$ ,  $i < g-1$  and thus by (4.11),  $y_j^2 = 0$  for all  $j$  since each monomial in  $Q_j$  involves some  $x_i$ ,  $i < g-1$  to a positive power. Thus  $q = (0, \dots, 0, x_{g-1}, 0, \dots, 0)$  and since we know the origin is in  $V$ , we can assume  $x_{g-1} \neq 0$ . Recall (3.8):  $t$  is a coordinate at  $\infty$  and  $dt = -\frac{1}{2} v(0) w_{g-1}$ ,  $v(0) \neq 0$ . Take  $\xi$  above  $(\infty, 0) \in R$  so that  $dt(\xi) = -\frac{1}{2} v(0) x_{g-1}$ . Then  $F(\xi) = q$ .

EXAMPLES. (4.15)  $g = 2$ .  $V \subset \mathbb{C}^3$  is a hypersurface defined by the single equation  $y^2 = Q(x_0, x_1)$ . In this case  $y^2 - Q$  generates the ideal of  $V$ .

(4.16)  $g = 3$ .  $V \subset \mathbb{C}^4$  is a complete intersection defined by the equations  $x_0 x_2 = x_1^2$ ,  $y_0^2 = Q_0(x_0, x_1, x_2)$ . More particularly, if  $R$  is given by  $u^2 = z^7 - 1$ , then the equations are  $x_0 x_2 = x_1^2$ ,  $y_0^2 = x_1 x_2^3 - x_0^4$ .

(4.17)  $g = 4$ .  $V \subset \mathbb{C}^6$  is given by the six equations

$$\begin{aligned} x_0 x_2 &= x_1^2, & x_1 x_3 &= x_2^2, & x_0 x_3 &= x_1 x_2, \\ y_1 x_0 &= y_0 x_1, & y_0^2 &= Q_0(x), & y_1^2 &= Q_1(x). \end{aligned}$$

In particular, if  $u^2 = z^9 - 1$ , then the last two equations become

$$y_0^2 = x_0 x_3^3 - x_0^4 = x_0(x_3^3 - x_0^3), \quad y_1^2 = x_2 x_3^3 - x_0^3 x_2 = x_2(x_3^3 - x_0^3).$$

We note that none of these equations can be eliminated.

### 5. Local invariants: the hyperelliptic case.

5.1. PROPOSITION. (i) ( $g = 2$ ). The tangent cone (not counting multiplicity) to  $V$  at 0 is the linear space  $y = 0$ .

(ii) ( $g > 2$ ). The tangent cone to  $V$  is given by the equations

$$\{x_0x_2 = x_1^2, x_1x_3 = x_2^2, \dots, x_{g-3}x_{g-1} = x_{g-2}^2, x_0x_{g-1} = x_1x_{g-2}, \\ y_0 = 0, \dots, y_{g-3} = 0\}. \quad (5.2)$$

PROOF. (i) is obvious.  $y^2$  is the initial form of the function  $y^2 - Q$  which generates the ideal of  $V$ .

(ii) Let  $W$  be the variety defined by the equations (5.2). Since these functions are all initial forms of functions vanishing on  $V$  (replace  $y_j$  by  $y_j^2$ ), the tangent cone  $CV$  is contained in  $W$ . On the other hand  $F: T \rightarrow V$  takes  $Z$  to the origin, so  $dF: T|Z \rightarrow CV$ . But  $dF = (F_0, 0)$  and the image of  $F_0$  is precisely the variety defined by the  $x$ -equations in (5.2). Thus  $W = dF(T|Z) \subset CV$ .

REMARK. Of course, if we knew that equations (4.6)–(4.9) generate the ideal of  $V$ , (ii) would also be obvious.

5.3. PROPOSITION. Let  $QV, QV_0$  be the quadratic transforms of  $V, V_0$  respectively.  $QV$  is a 2-sheeted branched cover of  $QV_0$  which is locally reducible along the exceptional variety (except at branch points). The exceptional variety of  $QV$  is a  $\mathbf{P}^1$ .

PROOF. Let  $\pi: Q \rightarrow C^{2g-2}$  be the quadratic transform of the origin in  $C^{2g-2}$ ,  $\pi_0: Q_0 \rightarrow C^{g-1}$  that for  $C^{g-1}$ . Then  $QV$  is the closure of  $\pi^{-1}(V)$  and  $QV_0$  is the closure of  $\pi_0^{-1}V_0$ . The map  $F: T \rightarrow C^{2g-2}$  lifts to  $\tilde{F}: T \rightarrow Q$ . For we can write  $\tilde{F}: T - Z \rightarrow Q$  in coordinates as

$$\tilde{F}(\xi) = (w(\xi), \tau(\xi), [w(\xi), \tau(\xi)]).$$

As  $\xi \rightarrow \xi_0 \in Z$ , some  $w_i(\xi_0)$  is nonzero (as a section of  $T^*$ ); take  $i = 0$ . Then, near  $\xi_0$

$$[w_0(\xi), \dots, w_{g-1}(\xi), \tau(\xi)] = [1, \dots, w_{g-1}(\xi)/w_0(\xi), \tau(\xi)/w_0(\xi)]$$

and the expression on the right extends  $\tilde{F}$ . Similarly  $F_0$  extends to  $\tilde{F}_0: T \rightarrow Q_0$ . Obviously, since  $F, \tilde{F}$  are surjective off  $z$ ,  $\tilde{F}(T) = QV$ ,  $\tilde{F}_0(T) = QV_0$ . Since one of  $w_0, \dots, w_{g-1}$  is nonzero at each point of  $Z_1$  and every coordinate of  $\tau$  vanishes to second order along  $Z$ , the map

$$((x, y), [x, y]) \rightarrow (x, [x]) \quad (5.4)$$

is well defined on  $QV$  and maps  $QV$  onto  $QV_0$  taking the exceptional set of  $QV$  one-to-one onto the exceptional set of  $QV_0$ . Otherwise (5.4) is 2 to 1, so it branches along the exceptional variety. It is easy to see this branching is reducible (generically), we verify it in the case  $g = 2$ .

$V$  is given by  $y^2 = Q(x_0, x_1)$ . Let  $[v_0, v_1, u]$  be homogeneous coordinates for  $\mathbf{P}^2$ ;  $QV$  is the closure of the graph of the map

$$v_0 = x_0, \quad v_1 = x_1, \quad y = u \quad (\text{not all of } x_0, x_1, u \text{ are zero}).$$

Let us write this in inhomogeneous coordinates on the coordinate neighborhood  $u_0 = \{v_1 \neq 0\}$ . Then  $\xi = v_1 v_0^{-1}$ ,  $\zeta = uv_0^{-1}$  are coordinates for  $U_0$  and in  $\mathbb{C}^3 \times U_0$ ,  $V$  is given by

$$y^2 = Q(x_0, x_1), \quad y = x_0 \zeta, \quad x_1 = x_0 \xi.$$

The first equation becomes

$$x_0^2 \zeta^2 = Q(x_0, x_1) = x_0^6 Q(1, \xi) \quad \text{or} \quad \zeta^2 = x_0^4 Q(1, \xi).$$

If  $Q(1, \xi) \neq 0$ , it has two square roots  $R_1(\xi)$ ,  $R_2(\xi)$ , and  $QV$  consists of the two nonsingular varieties coordinatized by  $x_0, \xi$  via the functions

$$\begin{aligned} \zeta &= x_0^2 R_i(\xi), \\ y &= x_0^3 R_i(\xi), \quad i = 1 \text{ or } 2. \\ x_1 &= x_0 \xi, \end{aligned}$$

**5.5. PROPOSITION.** *For  $g \geq 2$ , the multiplicity of  $V$  at 0 is  $\mu_0(V) = 2g - 2$ .*

**PROOF.** For  $g = 2$ ,  $y^2 - Q$  has degree  $2 = 2g - 2$ . Consider  $g > 2$ , and look at the tangent cone  $CV$  given by the equations (5.2). We use the technique of Whitney [8, Chapter 7, §7] for computing  $\mu_0(V)$ . Let  $\Pi_1 = \{x_0 = x_{g-1} = 0\}$  and

$$\Pi_2 = \{x_1 = \cdots = x_{g-2} = y_0 = \cdots = y_{g-3} = 0\}.$$

Since  $CV \cap \Pi_1 = \{0\}$ , if  $\psi$  is the projection onto  $\Pi_1$  with kernel  $\Pi_2$ ,  $\psi$  is (locally) proper on  $CV$ , and the multiplicity is the generic number of points in  $\psi^{-1}(q) \cap V$  for  $q \in \Pi_1$ . Let  $q = (a, b)$  with  $ab \neq 0$ . If

$$\psi(x_0, \dots, x_{g-1}, y_0, \dots, y_{g-3}) = q,$$

we must have

$$x_0 = a, \quad x_{g-1} = b, \quad x_1^{g-1} = a^{g-2}b \quad \text{and} \quad x_j = x_i^j / a^{j-1}, \quad j \neq 0, 1, g-1. \quad (5.6)$$

This system has  $g - 1$  different solutions. We show that each one gives rise to precisely two points in  $V$  so long as  $x_1 a^{-1}$  is not a root of  $Q_0(1, z, \dots, z^{g-1}) = 0$  (still a generic condition). Then at a point of  $V$  above an  $x$  satisfying

$$y_0^2 = a^4 Q_0(1, x_1 a^{-1}, \dots, (x_1 a^{-1})^{g-1}) \neq 0,$$

$y_0$  admits two distinct values. For each choice let

$$y_j = (x_j / x_0) y_0, \quad 1 \leq j \leq g-3.$$

Then  $(x_0, \dots, x_{g-1}, y_0, \dots, y_{g-1}) \in V$ , and these are the only possible choices. Since there are  $2(g-1)$  such choices,  $\mu_0 = 2(g-1)$ .

**5.7. PROPOSITION.** (i) *If  $g = 2$ , the minimal embedding dimension of  $V$  at 0 is 3.*

(ii) *For  $g > 2$  the minimal embedding dimension of  $V$  is  $2g - 2$ .*

*In other words  $V$  as constructed in §4 is minimally embedded.*

**PROOF.** (i) For  $g = 2$ , since  $V$  is singular, the minimal embedding dimension must be at least 3, and since it is attained, it is 3.

(ii) Now take  $g > 2$  and let

$$f(x, y) = \sum_{i=0}^{g-1} \lambda_i x_i + \sum_{j=0}^{g-3} \mu_j y_j + \sum a_{rs} x_r x_s + \text{other higher order terms}$$

be a function vanishing on  $V$ . Then  $f \circ F$  vanishes on  $T$ . If we write  $f \circ F$  as a series of homogeneous polynomials on  $T$ , each term must vanish independently on  $V$ . But the linear term is  $\sum_{i=0}^{g-1} \lambda_i \omega_i$ , so, since the  $\omega_i$  are independent,  $\lambda_0 = \cdots = \lambda_{g-1} = 0$ . The second-order term is

$$\sum_{j=0}^{g-3} \mu_j \tau_j + \sum a_{rs} \omega_r \omega_s.$$

Thus  $\sum \mu_j \tau_j \in S^2 H^0(K)$ , but (following the proof of Theorem 3.15(II)(i)) the  $\tau$ 's were chosen to be independent mod  $S^2 H^0(K)$ , so  $\mu_0 = \cdots = \mu_{g-3} = 0$  also. Thus if  $f$  vanishes on  $V$  we have  $df(0) \equiv 0$ , so the minimal embedding dimension is not less than  $2g - 2$ .

**6. The diagonal embedding.** For  $R$  any compact Riemann surface, let  $\Delta: R \rightarrow R \times R$  be the *diagonal embedding*.  $\Delta(p) = (p, p)$ . It is well known that the normal bundle to the diagonal is the tangent bundle  $TR$  to  $R$ . Thus if  $R$  has genus  $g \geq 2$ ,  $\Delta(R)$  is negatively embedded and  $R \times R$  can be blown down along the diagonal. Let  $\tau: R \times R \rightarrow W$  be the Grauert blowdown. In this section we shall be studying  $W$ , comparing it with the blowdown  $V$  of the zero-section of  $TR$ . In [4], we showed that these two embeddings are always first-order equivalent and are second-order equivalent if and only if  $g = 1$ . In the genus 1 case, they are in fact analytically equivalent.

**6.1. PROPOSITION.** *If  $R$  is an elliptic curve ( $g = 1$ ), then the embeddings  $z_T$  and  $\Delta$  are analytically equivalent.*

**PROOF.** Since  $g = 1$ ,  $TR$  is holomorphically trivial:  $TR = R \times \mathbb{C}$  and  $Z = R \times \{0\}$ . Let  $\nu: \mathbb{C} \rightarrow R$  be the universal covering of  $R$ . Define a map  $\Theta: R \times \mathbb{C} \rightarrow R \times R$  by

$$\Theta(\nu(x), y) = (\nu(x), \nu(x + y)).$$

$\Theta$  is obviously locally one-to-one, and has differential of maximal rank,

$$d\Theta = \begin{pmatrix} d\nu & 0 \\ d\nu & d\nu \end{pmatrix}.$$

$\Theta$  maps  $R \times \{0\}$  one-to-one onto  $\Delta(R)$ , so  $\Theta$  must be biholomorphic in a neighborhood of  $R \times \{0\}$ , thus providing the necessary equivalence.

Now we restrict attention to the case of genus  $g \geq 2$ . Let  $R$  be a compact Riemann surface of genus  $g \geq 2$ . Let  $j: R \rightarrow X$  be an embedding of  $R$  in a two-dimensional manifold  $X$  with normal bundle the tangent bundle  $TR$ . We shall henceforth call this *the standard situation*. Let  $\pi: X \rightarrow Y$  be the Grauert blowdown of  $R$  to a point, denoted 0.  $\mathcal{I}$  is the ideal sheaf of  $R$  in  $X$ , and  $\mathfrak{m}$  the maximal ideal at 0.

6.2. LEMMA.  $H^1(R, \mathcal{G}^k) = 0$  for  $k \geq 2$ .

PROOF. If  $U, U'$  are two strictly pseudoconvex neighborhoods of  $R$ , then

$$H^v(U, \mathcal{F}) = H^v(U', \mathcal{F}), \quad \text{all } v \geq 1, \quad (6.3)$$

for any coherent sheaf  $\mathcal{F}$ . Since  $R$  has a neighborhood basis of strictly pseudoconvex domains, it suffices to show  $H^1(U, \mathcal{G}^k) = 0$  for  $U$  such a domain and  $k \geq 2$ . We have the exact sequence

$$\cdots \rightarrow H^1(U, \mathcal{G}^{j+1}) \xrightarrow{\alpha_j} H^1(U, \mathcal{G}^j) \rightarrow H^1(R, K^j) \rightarrow \cdots$$

If  $j \geq 2$ ,  $H^1(R, \mathcal{G}^j) = 0$ , so the map  $\alpha_j$  is surjective for all  $j \geq 2$ . Composing such maps, we find that all maps

$$H^1(R, \mathcal{G}^{j+k}) \rightarrow H^1(R, \mathcal{G}^k), \quad j \geq 0, k \geq 2, \quad (6.4)$$

are surjective. But, by [2, §4, Satz 1], for any  $k$ , there is a  $j$  sufficiently large so that (6.3) is the zero map. So (6.3) is both surjective and identically zero; thus  $H^1(R, \mathcal{G}^k) = 0$ ,  $k \geq 2$ .

6.5. PROPOSITION. In the standard situation,  $\mathcal{G}^k = \hat{\mathfrak{m}}^k$ .

PROOF. For  $p \in R \subset X$ , we have to show that  $\mathcal{O}_p \pi^{-1} \mathfrak{m}_0^k = \mathcal{G}_p^k$ . Clearly  $\mathcal{O}_p \pi^{-1} \mathfrak{m}_0^k \subset \mathcal{G}_p^k$ . By Theorem 1.6 there is an  $\omega \in H^0(R, K)$  with  $\omega(p) \neq 0$ . By Lemma 6.2 the map  $\beta$  in the exact sequence

$$\cdots \rightarrow H^0(U, \mathcal{G}) \xrightarrow{\beta} H^0(R, K) \rightarrow H^1(U, \mathcal{G}^2) \rightarrow \cdots$$

is surjective, so there is an  $f \in H^0(U, \mathcal{G}) = \mathfrak{m}_0$  with  $\beta f = \omega$ . Then  $f$  vanishes to order exactly one at  $p$ , so for all  $k$ ,  $f^k$  generates  $\mathcal{G}_p^k$  over  $\mathcal{O}_p$ . But  $f^k \in \pi^{-1}(\mathfrak{m}_0^k)$ .

6.6. PROPOSITION. In the standard situation, suppose  $R$  is hyperelliptic of genus  $g > 2$ . Then the natural injection

$$\mathfrak{m}_0^2 \rightarrow H^0(R, \mathcal{G}^2) \quad (6.7)$$

is not surjective.

PROOF. Suppose (6.7) were surjective. Let  $\sigma \in H^0(R, K^2)$ . By Lemma 6.2 ( $k = 3$ ), there is  $f \in H^0(R, \mathcal{G}^2)$  whose initial term (in the Taylor expansion along  $R$ ) is  $\sigma$ . If (6.7) were surjective, we could write  $f = \sum a_{ij} g_i g_j$  with  $g_i \in \mathfrak{m}_0 = H^0(R, \mathcal{G})$ . Letting  $\sigma(g_i) \in H^0(R, K)$  be the initial term of the Taylor expansion of  $g_i$ , we have

$$\sigma = \sigma(f) = \sum a_{ij} \sigma(g_i) \sigma(g_j),$$

so that  $\sigma$  is in the image of  $S^2 H^0(R, K)$ . But the map  $S^2 H^0(R, K) \rightarrow H^0(R, K^2)$  is not surjective (Theorem 3.15(II)(i)).

Similarly, by using Theorem 3.15(I)(ii) we obtain

6.8. PROPOSITION. In the standard situation, if  $g = 2$ , the natural injection  $\mathfrak{m}_0^3 \rightarrow H^0(R, \mathcal{G}^3)$  is not surjective.

REMARK. In the general situation of the Grauert blowdown  $X \rightarrow \tilde{X}$  we shall always have (using Theorem 1.3)  $H^0(R, \mathcal{G}^k) \subset H^0(R, \mathcal{G})^{F(k)}$  where  $F(k) \rightarrow \infty$  as  $k \rightarrow \infty$ . We shall make this more precise for the standard situation being considered here.

6.9. PROPOSITION. *In the standard situation, if  $R$  is not hyperelliptic, the natural injection  $m_0^k \rightarrow H^0(R, \mathcal{G}^k)$  is an isomorphism for all  $k$ .*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & m_0^k & \xrightarrow{\alpha} & H^0(R, \mathcal{G}^{k+1}) \\
 \alpha_1 \nearrow & & \searrow \alpha_2 & & \downarrow \varepsilon \\
 S^k m_0 & \xrightarrow{\alpha} & & & H^0(R, \mathcal{G}^k) \\
 \downarrow \delta & & & & \downarrow \gamma \\
 S^k H^0(R, K) & \xrightarrow{\beta} & & & H^0(R, K^k) \\
 \downarrow & & & & \downarrow \\
 0 & & & & 0
 \end{array} \quad (6.10)$$

The maps  $\delta$ ,  $\gamma$ ,  $\alpha_1$  are surjective;  $\alpha_2$  and  $\varepsilon$  are injective. By Noether's theorem,  $\beta$  is surjective.

Let  $f \in H^0(R, \mathcal{G}^k)$ . Then  $\gamma(f) = \beta\delta\Theta$  for some  $\Theta \in S^k m_0$ . By commutativity,  $\gamma(f - \alpha\Theta) = 0$ , so  $f - \alpha\Theta = g$  for some  $g \in H^0(R, \mathcal{G}^{k+1})$ . Thus

$$H^0(R, \mathcal{G}^k) \subset m_0^k + H^0(R, \mathcal{G}^{k+1}).$$

Proceeding inductively for  $j > k$  (since  $m_0^j \subset m_0^k$ ), we have

$$H^0(R, \mathcal{G}^k) \subset m_0^k + H^0(R, \mathcal{G}^j) \subset m_0^k + m_0^{F(j)}, \quad \text{all } j > k,$$

by the preceding remark. Since  $F(j) \rightarrow \infty$ , eventually  $F(j) > k$ , thus  $H^0(R, \mathcal{G}^k) \subset m_0^k$ , proving the proposition.

In the hyperelliptic case the best we can do is

6.11. PROPOSITION. *In the standard situation with  $R$  hyperelliptic, we have*

(i)  $g = 2$ :

$$H^0(R, \mathcal{G}^{k+1}) \subset H^0(R, \mathcal{G}^3)m_0^{k-2} \subset m_0^{k-1}, \quad k > 3,$$

(ii)  $g > 2$ :

$$H^0(R, \mathcal{G}^{k+1}) \subset H^0(R, \mathcal{G}^2)m_0^{k-1} \subset m_0^k, \quad k > 2.$$

PROOF. We shall prove (i); the proof of (ii) is similar. Diagram (6.10) has to be replaced by

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & H^0(K, \mathcal{G}^3)m_0^k & \xrightarrow{\alpha} & H^0(R, \mathcal{G}^{k+4}) \\
 \alpha_1 \nearrow & & \searrow \alpha_2 & & \downarrow \varepsilon \\
 S^k m_0 \times H^0(R, \mathcal{G}^3) & \xrightarrow{\alpha} & & & H^0(R, \mathcal{G}^{k+3}) \\
 \searrow \delta & & & & \downarrow \\
 S^k H^0(R, K) \otimes H^0(R, K^3) & \xrightarrow{\beta} & & & H^0(R, K^{k+3}) \\
 \downarrow & & & & \downarrow \\
 0 & & & & 0
 \end{array} \quad (6.12)$$



and the proof is now the same as that of Proposition 6.9 ( $\beta$  is surjective now because of Theorem 3.15(I)(iv)).

Let  $q$  be a point in an analytic space  $Y$  with structure sheaf  $\mathcal{O}$ . Let  $\mathfrak{m}$  be the maximal ideal at  $q$ . The associated graded ring at  $q$  is

$$\mathrm{gr}_{\mathfrak{m}}(\mathcal{O}_q) = \bigoplus_{k=0}^{\infty} \mathfrak{m}^k / \mathfrak{m}^{k+1}.$$

This is a homogeneous ring which is in fact the ring of polynomials on the tangent cone  $C_q Y$  of  $Y$  at  $q$ . The dimensions of the finite-dimensional vector spaces  $\mathfrak{m}^k / \mathfrak{m}^{k+1}$  determine the Hilbert polynomial of  $\mathcal{O}_q$  and hence the multiplicity of  $Y$  at  $q$ . The dimension of  $\mathfrak{m} / \mathfrak{m}^2$  is the minimal embedding dimension of  $Y$  at  $q$ .

6.13. THEOREM. *In the standard situation suppose  $R$  is nonhyperelliptic of genus  $g > 2$ . Then*

$$\mathrm{gr}_{\mathfrak{m}}(\mathcal{O}_0) \cong \bigoplus_{k=0}^{\infty} H^0(R, K^k) \quad (6.14)$$

as graded rings.

PROOF. By Proposition 6.9,

$$\mathfrak{m}^k / \mathfrak{m}^{k+1} \cong H^0(R, \mathcal{G}^k) / H^0(R, \mathcal{G}^{k+1}),$$

and by Lemma 6.2

$$H^0(R, K^k) \cong H^0(R, \mathcal{G}^k) / H^0(R, \mathcal{G}^{k+1}).$$

Thus (6.14) is true as graded vector spaces. The map  $\mathfrak{m}^k / \mathfrak{m}^{k+1} \rightarrow H^0(R, K^k)$  is induced by the map  $\mathfrak{m}^k \rightarrow H^0(R, K^k)$  taking an  $f \in \mathfrak{m}_0^k$  to the initial term  $\sigma(f)$  in the Taylor expansion of  $f \circ \pi$  along  $R$ . Since  $\sigma(fg) = \sigma(f)\sigma(g)$ , (6.14) is also a ring isomorphism.

6.15. THEOREM. *Let  $R$  be nonhyperelliptic of genus  $g > 2$ . Let  $R \rightarrow X$  be an embedding of  $R$  in a smooth surface so that the normal bundle is the tangent bundle of  $R$ . Let  $\pi: X \rightarrow \tilde{X}$  be the Grauert blowdown of  $R$  to  $Q$  and  $\tau: TR \rightarrow V$  the Grauert blowdown of the normal bundle. Then*

- (a)  $\pi: X \rightarrow \tilde{X}$  is a monoidal transformation,
- (b)  $V$  is the tangent cone  $C_0(\tilde{X})$ .

PROOF. (a) Follows from [4, Theorem 5.7] and (b) is a corollary of Theorem 6.13.

In particular, in the nonhyperelliptic case, the Grauert blowdown of the diagonal in  $R \times R$  has the same minimal embedding dimension, Hilbert polynomial and multiplicity on the blowdown of the zero section of the tangent bundle.

6.16. THEOREM. *In the standard situation, suppose  $R$  is hyperelliptic of genus  $g > 2$ . Then, for all  $k$ ,  $\mathfrak{m}^k / \mathfrak{m}^{k+1}$  is independent of the embedding (i.e., is the same as for the embedding in the tangent bundle).*

PROOF. For each  $k \geq 0$ , let  $I_k = \prod_{j \geq k} H^0(R, K^j)$ . Then  $I_0$  is the ring of formal power series in the fiber coordinate of  $TR$ , and the Taylor expansion provides a

map  $H^0(Z, \mathcal{O}) \rightarrow I_0$  which takes  $H^0(R, \mathcal{G}^k) \rightarrow I_k$  (where  $\mathcal{G}$  is the ideal sheaf of the zero section of  $TR$ ). Since

$$H^0(R, \mathcal{G}^{k+2}) \subset \mathfrak{m}_0^{k+1} \subset \mathfrak{m}_0^k \subset H^0(R, \mathcal{G}^k),$$

the map  $H^0(R, \mathcal{G}^k) \rightarrow I_k$  induces an injection

$$\mathfrak{m}_0^k / \mathfrak{m}_0^{k+1} \rightarrow I_k / I_{k+2} = H^0(R, K^k) + H^0(R, K^{k+1}). \quad (6.17)$$

Let  $R \rightarrow X$  be another embedding in the standard situation, with  $Y$  the Grauert blowdown,  $\mathfrak{m}_Y$  the maximal ideal at the singularity, and  $\mathcal{G}_Y$  the ideal sheaf of  $R$  in  $X$ . Let  $\sigma_k$  be the natural map

$$\sigma_k: H^0(R, \mathcal{G}_Y^k) \rightarrow H^0(R, K^k).$$

Since  $\sigma_k$  is surjective, there is a map  $\omega_k: H^0(R, K^k) \rightarrow H^0(R, \mathcal{G}_Y^k)$  such that  $\sigma_k \circ \omega_k = \text{identity}$ . Define

$$\tau_k: H^0(R, \mathcal{G}_Y^k) \rightarrow I_k / I_{k+2}$$

by

$$\tau_k(f) = (\sigma_k(f), \sigma_{k+1}(f - \omega_k \sigma_k f)).$$

We shall show that  $\tau_k|_{\mathfrak{m}_Y^k}$  induces an isomorphism  $\tilde{\tau}_k$  of  $\mathfrak{m}_Y^k / \mathfrak{m}_Y^{k+1}$  with the image  $R^k$  of  $\mathfrak{m}_0^k / \mathfrak{m}_0^{k+1}$  under (6.17).

(a) Let  $f \in \mathfrak{m}_Y^k, f = \sum f_{i_1} \dots f_{i_k}$  with  $\{f_{i_j}\} \subset \mathfrak{m}_Y$ . Then

$$\sigma_k(f) = \sum \sigma_1(f_{i_1}) \dots \sigma_1(f_{i_k}) \in \mathfrak{m}_0^k,$$

and  $\sigma_k(f - \omega_k \sigma_k f) = 0$ . Thus, the initial form of  $f - \omega_k \sigma_k f$  is  $\sigma_\nu(f - \omega_k \sigma_k f)$ ,  $\nu > k$ . As a section of  $K^\nu$  it is in  $H^0(R, \mathcal{G}^\nu)$ , where this group is associated to the canonical embedding  $R \rightarrow TR$ . Applying Proposition 6.11(ii) to this embedding, we obtain  $\sigma_{k+1}(f - \omega_k \sigma_k f) \in \mathfrak{m}_0^k$  (if  $\nu = k + 1$ ; if  $\nu > k + 1$  this is 0, so certainly in  $\mathfrak{m}_0^k$ ). Thus  $\tau_k(f) \in \mathfrak{m}_0^k$  and the map  $\tilde{\tau}_k$  is well defined and has image in  $R^k$ .

(b)  $\tau_k$  is injective. If  $\tau_k(f) = 0$ , then  $\sigma_k(f) = 0$  and  $\sigma_{k+1}(f) = 0$ , so  $f \in H^0(R, \mathcal{G}_Y^{k+2}) \subset \mathfrak{m}_Y^{k+1}$ .

(c)  $\tau_k$  is surjective. Let  $g = (g_0, g_1) \in \mathfrak{m}_0^k / \mathfrak{m}_0^{k+1}$ . Then  $g_0 = \sum g_{i_1} \dots g_{i_k}$  with  $g_{i_j} \in \mathfrak{m}_0$ . Let

$$f = \sum \omega_1 \sigma_1(g_{i_1}) \dots \omega_k \sigma_k(g_{i_k}).$$

Then  $f \in \mathfrak{m}_Y^k$  and  $\sigma_k(f) = g_0$ . Let  $\tau_k(f) = (g_0, g'_1)$ . Choose  $f' \in H^0(R, \mathcal{G}_Y^{k+1})$  such that  $\sigma_{k+1}(f') = g_1 - g'_1$ . Then  $f + f' \in \mathfrak{m}_Y^k$  and  $\tau_k(f + f') = (g_0, g_1)$ .

**6.18. COROLLARY.** *Let  $R$  be hyperelliptic with genus  $g > 2$ . Let  $R$  be embedded in  $X$  with normal bundle  $TR$ . Then the Hilbert polynomial of the Grauert blowdown is independent of the embedding, as is the minimal embedding dimension. In particular the blowdown of  $\Delta \subset R \times R$  and  $Z \subset TR$  have the same multiplicity and minimal embedding dimension at the singular point.*

**REMARK.** Since nothing is said of the multiplicative structure of the ring  $\text{gr}(\mathcal{O}_Y)$ , we cannot conclude that all such blowdowns have the same tangent cone. Furthermore, the above argument breaks down for  $g = 2$ , and we do not know whether or not the same result holds.

## BIBLIOGRAPHY

1. C. Banica and O. Stanasila, *Algebraic methods in the global theory of complex spaces*, Wiley, New York, 1976.
2. H. Grauert, *Über modifikationen und exceptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.
3. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.
4. J. Morrow and H. Rossi, *Some general results on equivalence of embeddings*, Proceedings of Conference on Complex Analysis, Princeton, N. J., 1979 (to appear).
5. M. Noether, *Über die invariante Darstellung algebraischer Funktionen*, Math. Ann. **17** (1880), 263–284.
6. K. Petri, *Über die invariante Darstellung algebraischer Funktionen einer Veränderlichen*, Math. Ann. **88** (1922), 242–289.
7. B. Saint-Donat, *On Petri's analysis of the linear system of quadratics through a canonical curve*, Math. Ann. **206** (1973), 157–175.
8. H. Whitney, *Complex analytic varieties*, Addison-Wesley, Reading, Mass., 1972.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112